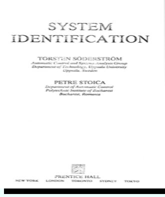
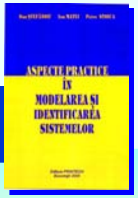
❺ Exerciţii rezolvate



Exerciţiul 1.1



Fie *e* un zgomot alb de medie nulă şi varianţă unitară care stimulează intrarea unui model AR[1]. Să se determine secvenţa de auto-covarianţă a zgomotului colorat rezultat, *y*.

Soluţie

$$\xrightarrow{E} \Big| \ast y[n-k] \Big| \quad y[n] + a y[n-1] = e[n] \Rightarrow r_y[k] + a r_y[k-1] = r_{ey}[k], \ \forall k \geq 0$$

$$r_{ey}[k] = E\{e[n]\, y[n-k]\} = E\left\{ e[n] \frac{1}{A(q^{-1})} e[n-k] \right\} =$$

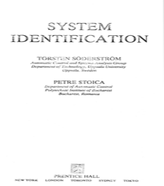
$$= E\left\{ e[n] \sum_{m\geq 0} \alpha_m e[n-k-m] \right\} = \sum_{m\geq 0} \alpha_m r_e[k+m] =$$

$$= \sum_{m\geq 0} \alpha_m \delta_0[k+m] = \alpha_{-k}, \quad \forall k \geq 0$$

$$r_{ey}[k] = \alpha_0 \delta_0[k] = \delta_0[k]$$

$$\boxed{\frac{1}{1+aq^{-1}} = \underbrace{1}_{\alpha_0} \underbrace{-a}_{\alpha_1} q^{-1} + \alpha_2 q^{-2} + \dots}$$

❺ Exerciţii rezolvate



Soluţie (Exerciţiul 1.1)

$$r_y[k] + a\, r_y[k-1] = \delta_0[k]$$

$$\begin{aligned} k=0: &\quad \begin{cases} r_y[0] + a\, r_y[1] = 1 \\ r_y[1] + a\, r_y[0] = 0 \end{cases} \\ k=1: & \end{aligned}$$

$$r_y[0](1-a^2) = 1 \Rightarrow r_y[0] = \frac{1}{1-a^2}$$

$$r_y[1] = -\frac{a}{1-a^2}$$

$$\begin{aligned} k=1: &\quad \cancel{r_y[1]} + a\, r_y[0] = 0 & \left(-\tfrac{1}{a}\right) \\ k=2: &\quad \cancel{r_y[2]} + \cancel{a\, r_y[1]} = 0 & \left(-\tfrac{1}{a}\right)^2 \\ &\quad \vdots & \\ k=k: &\quad r_y[k] + \cancel{a\, r_y[k-1]} = 0 & \left(-\tfrac{1}{a}\right)^{k-1} \end{aligned}$$

$$r_y[k]\left(-\frac{1}{a}\right)^{k-1} = -a\, r_y[0].$$

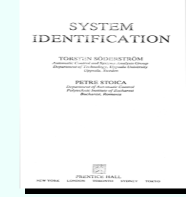
$$\Downarrow$$

$$r_y[k] = (-a)^k \frac{1}{1-a^2} = \frac{(-a)^k}{1-a^2} \qquad \forall k \geq 1$$

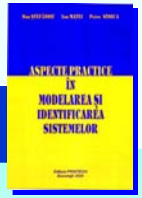
$$\boxed{r_y[k] = \frac{(-a)^{|k|}}{1-a^2} \qquad |a| \neq 1 \qquad \forall k \in \mathbb{Z}}$$

✸ Folosind relaţia analitică a secvenţei de auto-covarianţă, se poate determina parametrul necunoscut a din datele măsurate (după estimarea auto-covarianţei).

❺ Exerciții rezolvate



Exercițiul 1.2



Fie *e* un zgomot alb de medie nulă şi varianță unitară care stimulează intrarea unui model MA[1]. Să se determine secvența de auto-covarianță a zgomotului colorat rezultat, *y*. Dacă modelul MA are ordinul *nc*, să se arate că secvența de auto-covarianță a ieşirii are suport finit (adică are un număr finit de valori nenule) şi să se determine dimensiunea maximă a suportului.

Soluție

ⓐ

$$r_y[k] = r_{ey}[k] + c\, r_{ey}[k-1], \quad \forall k \geq 0$$

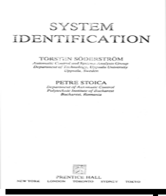
$$r_{ey}[k] = E\{e[n]\, y[n-k]\} = E\{e[n]\,[e[n-k] + c\, e[n-k-1]]\} =$$

$$= r_e[k] + c\, r_e[k+1], \quad \forall k \geq 0$$

$$r_{ey}[k] = \delta_0[k], \ \forall k \geq 0; \quad r_{ey}[-1] = c; \quad r_{ey}[-2] = 0, \quad r_{ey}[k] = 0 \ \forall k \leq -2$$

$$\begin{cases} r_y[0] = 1 + c^2 \\ r_y[1] = c \\ r_y[k] = 0, \ \forall k \geq 2 \end{cases}$$

❺ Exerciții rezolvate



Soluție (Exercițiul 1.2)

(b) $MA[nc]: \; y[n] = e[n] + c_1 e[n-1] + \dots + c_{nc} e[n-nc], \; \forall n \geq 0$

$$\Downarrow$$

$$r_y[k] = r_{ey}[k] + c_1 r_{ey}[k-1] + \dots + c_{nc} r_{ey}[k-nc], \; \forall k \geq 0$$

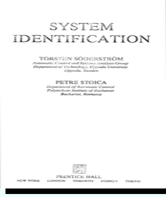
$$r_{ey}[k] = E\{e[n]\,y[n-k]\} = E\{e[n]\,[e[n-k] + c_1 e[n-k-1] + \dots + c_{nc}\, e[n-k-nc]]\}$$

$$= \delta_0[k] + c_1 \delta_0[k+1] + \dots + c_{nc} \delta_0[k+nc], \; \forall k \in \mathbb{Z}.$$

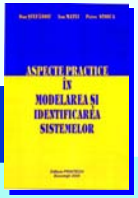
$$\begin{cases} r_y[0] = 1 + c_1^2 + \dots + c_{nc}^2 \\ r_y[1] = c_1 + c_1 c_2 + \dots + c_{nc-1} c_{nc} \\ r_y[2] = c_2 + c_1 c_3 + \dots + c_{nc-2} c_{nc} \\ \vdots \\ r_y[nc] = c_{nc} \\ r_y[nc+1] = 0, \; r_y[k] = 0, \; \forall k \geq nc+1 \end{cases}$$

$$\mathrm{Supp}(r_y) = \overline{-nc, +nc}$$

❺ Exerciții rezolvate



Exercițiul 1.3



Prin filtrarea unui zgomot alb *e* de medie nulă şi varianță unitară se obține un zgomot colorat *y* cu densitatea spectrală:

$$\phi_y(\omega) = \frac{0.75}{1.25 - \cos\omega}, \quad \forall \omega \in \mathbb{R}.$$

Considerînd că filtrul utilizat are funcția de sistem:

$$H(q^{-1}) = \frac{b_1 q^{-1}}{1 + a_1 q^{-1}},$$

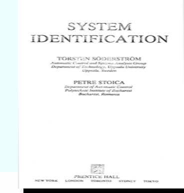
să se determine cei 2 parametri ai acestuia. Pot fi ei determinați în mod unic folosind numai analiza spectrală? Evaluați de asemenea varianța zgomotului colorat.

Soluție

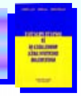
$$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot \overline{H(e^{j\omega})} = \frac{b_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}} \cdot \frac{b_1 e^{j\omega}}{(1 + a_1 e^{j\omega})} =$$

$$= \frac{b_1^2}{1 + 2a_1\cos\omega + a_1^2} = \frac{0.75}{(1 + a_1^2) + 2a_1\cos\omega}$$

❺ Exerciții rezolvate



Soluție (Exercițiul 1.3)



$$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot \overline{H(e^{j\omega})} = \frac{b_1 e^{j\omega}}{1+a_1 e^{j\omega}} \cdot \overline{\frac{b_1 e^{j\omega}}{(1+a_1 e^{j\omega})}} =$$

$$= \frac{b_1^2}{1+2a_1\cos\omega + a_1^2} = \frac{0.75}{(1+a_1^2)+2a_1\cos\omega}$$

$$\Downarrow$$

$$\begin{cases} b_1 = \pm\sqrt{0.75} \\ 1+a_1^2 = 1.25 \Rightarrow a_1^2 = 0.25 \Rightarrow a_1 = \pm 0.5 \\ 2a_1 = -1 \Rightarrow a_1 = -0.5 \end{cases}$$

$$\Downarrow$$

$$a_1 = -\frac{1}{2}, \quad b_1 \in \{-\sqrt{0.75}, +\sqrt{0.75}\}$$

$\boxed{\text{Neunic!}}$

$$q^{-1}H(q^{-1}) = \frac{b_1}{1+aq^{-1}} = b_1 \sum_{k\geq 0} \alpha_k q^{-k}$$

$$1 \equiv \sum_{k\geq 0} \alpha_k q^{-k} + a_1 \sum_{k\geq 0} \alpha_k q^{-k-1} \equiv \sum_{k\geq 0} \alpha_k q^{-k} + a_1 \sum_{k\geq 1} \alpha_{k-1} q^{-k} \equiv$$

$$\equiv \alpha_0 + \sum_{k\geq 1} (\alpha_k + a_1\alpha_{k-1}) q^{-k}$$

$$\Rightarrow y[n] = \left[b_1 \sum_{k\geq 0} (-a_1)^k q^{-k-1}\right] e[n].$$

$$\begin{cases} \alpha_0 = 1 \\ \alpha_k = -a_1\alpha_{k-1}, \ \forall k \geq 1 \end{cases} \Rightarrow \alpha_k = (-a_1)^k, \ \forall k \geq 1$$

$$\sigma_y^2 = r_y[0] = b_1^2 \sum_{k\geq 0} a_1^{2k} = \frac{b_1^2}{1-a_1^2} = \frac{0.75}{1-0.25} = 1$$

☺