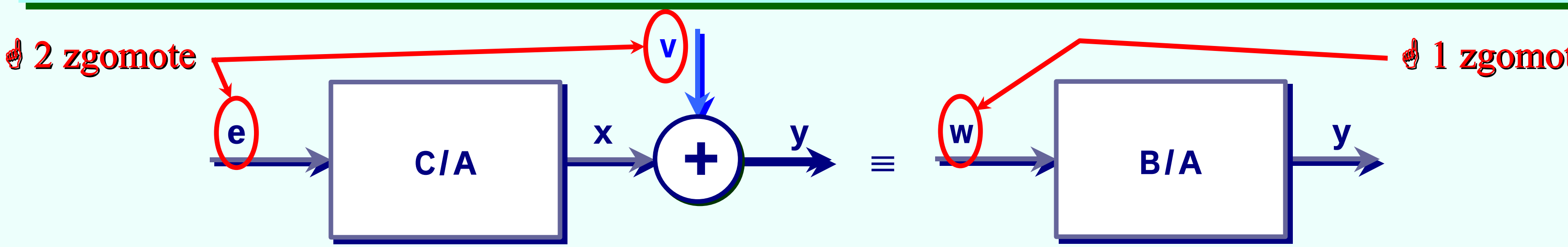
❺ Exerciții rezolvate

Exercițiul 1.4

Echivalența dintre 2 modele matematice (densități spectrale de ieşire identice).



$$\text{ARMA}[na,nc]:\;\; A(q^{-1})x[n] = C(q^{-1})e[n],\;\; \forall\, n\in\mathbb{N}^*$$

$$y[n] = x[n] + v[n],\;\; \forall\, n\in\mathbb{N}^*$$

$$E\{e[n]e[n\pm k]\} = \lambda_e^2\delta_0[k],\;\; \forall k\in\mathbb{Z}$$ (alb)

$$E\{v[n]v[n\pm k]\} = \lambda_v^2\delta_0[k],\;\; \forall k\in\mathbb{Z}$$ (alb)

$$E\{e[n]v[n\pm k]\} = 0,\;\; \forall k\in\mathbb{Z}$$ (necorelate)

$$\text{ARMA}[na,nb]:\;\; y[n] = \frac{B(q^{-1})}{A(q^{-1})}w[n],\;\; \forall\, n\in\mathbb{N}^*$$

$$E\{w[n]w[n\pm k]\} = \lambda_w^2\delta_0[k],\;\; \forall k\in\mathbb{Z}$$ (alb)

Să se determine coeficienții şi gradul polinomului necunoscut *B*, precum şi varianța λ_w^2 în funcție de polinoamele *A*, *C* şi varianțele λ_e^2, λ_v^2, prin echivalarea celor două modele, în cazul *na=nc=1*. Este modelul rezultat unic determinat?

☝ **Examen:**

Generalizați rezultatul pentru valori arbitrare ale indicilor structurali *na* şi *nc*.

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Soluţie (Exerciţiul 1.4)

densităţi spectrale de ieşire identice

⇔ autocovarianţe de ieşire identice

$\boxed{na = nc = 1}$

- zgomote :

— evident: $\underbrace{E\{y[n]y[n-k]\}}_{r_y[k]} = r_x[k] + \lambda_v^2 \delta_0[k]$, $\forall k \geq 0$

$$r_y[k] = r_x[k] + \lambda_v^2 \delta_0[k]$$

$$r_x[k] + a_1 r_x[k-1] = r_{ex}[k] + c_1 r_{ex}[k-1], \quad \forall k \geq 0$$

$$r_{ex}[k] = E\{e[n]x[n-k]\} = E\Big\{e[n] \sum_{m\geq 0} \alpha_0 e[n-m-k]\Big\} =$$

$$C(q^{-1}) \,\Big|\, \frac{A(q^{-1})}{\alpha_0 + \alpha_1 q^{-1} + \dots}, \quad \alpha_0 = 1$$

$$= \sum_{m \geq 0} \alpha_0 \lambda_e^2 \delta_0[m+k] = \lambda_e^2 \delta_0[k].$$

$$\Downarrow$$

$$r_x[k] + a_1 r_x[k-1] = \lambda_e^2 \big(\delta_0[k] + c_1 \delta_0[k-1]\big), \quad \forall k \geq 0$$

$\underline{k=0}$: $r_x[0] + a_1 r_x[1] = \lambda_e^2$

$\underline{k=1}$: $r_x[1] + a_1 r_x[0] = \lambda_e^2 c_1 \quad \big| \cdot (-a_1)$

$$(1-a_1^2)\, r_x[0] = \lambda_e^2 (1 - a_1 c_1) \Leftrightarrow \boxed{r_x[0] = \lambda_e^2 \frac{1-a_1c_1}{1-a_1^2}}$$

$$r_x[1] = \lambda_e^2 \frac{c_1 - a_1}{1-a_1^2}$$

❺ Exerciții rezolvate

Soluție (Exercițiul 1.4)

$$
\begin{array}{lr|l}
\cancel{r_x[1]} + a_1 r_x[0] = \lambda_e^2 c_1 & & 1 \\
\cancel{r_x[2]} + a_1 \cancel{r_x[1]} = 0 & & (-a_1)^{-1} \\
\vdots & & \vdots \\
r_x[k] + a_1 \cancel{r_x[k-1]} = 0 & & (-a_1)^{1-k} \\
\hline
\end{array}
$$

$$(-a_1)^{1-k} r_x[k] = \lambda_e^2 c_1 - a_1 r_x[0] = r_x[1] = \lambda_e^2 \frac{c_1 - a_1}{1 - a_1^2}$$

$$\Updownarrow$$

$$\boxed{r_x[k] = \frac{c_1 - a_1}{1 - a_1^2} \lambda_e^2 (-a_1)^{k-1}, \quad \forall k \geq 1}$$

$$\boxed{\begin{aligned} r_y[0] &= \frac{1 - a_1 c_1}{1 - a_1^2} \lambda_e^2 + \lambda_v^2 \\ r_y[k] &= (-a_1)^{k-1} \frac{c_1 - a_1}{1 - a_1^2} \lambda_e^2, \quad \forall k \geq 1 \end{aligned}}$$

❺ Exerciții rezolvate

Soluție (Exercițiul 1.4)

- 1 zgomot:

b_0

$$r_y[k] + a_1 r_y[k-1] = \left(b_0\delta_0[k] + b_1\delta_0[k-1] + \dots + b_{nb}\delta_0[k-nb]\right)\lambda_w^2 \quad \forall k\ge 0$$

$k=0$: $r_y[0] + a_1 r_y[1] = b_0^2\lambda_w^2$

$k=1$: $r_y[1] + a_1 r_y[0] = b_0 b_1\lambda_w^2 \quad |(-a_1)$

$$(1-a_1^2)\, r_y[0] = (b_0 - a_1 b_1)\lambda_w^2 \Leftrightarrow r_y[0] = b_0\frac{b_0-a_1b_1}{1-a_1^2}\lambda_w^2$$

b_0

$$r_y[1] = b_0\frac{b_1 - \cancel{a_1^2 b_1} - a_1 b_0 + \cancel{a_1^2 b_1}}{1-a_1^2}\lambda_w^2 = b_0\frac{b_1-a_1b_0}{1-a_1^2}\lambda_w^2$$

$$\begin{array}{l|l}
r_y[1] + a_1 r_y[0] = b_0 b_1\lambda_w^2 & \\
r_y[2] + a_1 r_y[1] = b_0 b_2\lambda_w^2 & (-a_1)^{-1} \\
\vdots & \\
r_y[nb] + a_1 r_y[nb-1] = b_0 b_{nb}\lambda_w^2 & (-a_1)^{1-nb} \\
r_y[nb+1] + a_1 r_y[nb] = 0 & (-a_1)^{-nb} \\
\vdots & \vdots \\
r_y[k] + a_1 r_y[k-1] = 0 & (-a_1)^{1-k}
\end{array}$$

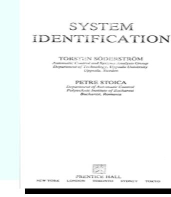
$$(-a_1)^{1-k} r_y[k] = -a_1 r_y[0] + \lambda_w^2 b_0 \sum_{m=1}^{\min\{k,nb\}} b_m (-a_1)^{1-m} \Leftrightarrow$$

$$\Leftrightarrow r_y[k] = (-a_1)^k r_y[0] + \lambda_w^2 b_0 \sum_{m=1}^{\min\{k,nb\}} b_m (-a_1)^{k-m} =$$

$$= (-a_1)^k b_0 \frac{b_0-a_1b_1}{1-a_1^2}\lambda_w^2 + \lambda_w^2 b_0 \sum_{m=1}^{\min\{k,nb\}} b_m (-a_1)^{k-m}, \quad \forall k\ge 1$$

❺ Exerciții rezolvate

Soluție (Exercițiul 1.4)



• echivalență

$\underline{k=0}$: $$\frac{1-a_1c_1}{1-a_1^2}\lambda_e^2 + \lambda_v^2 = b_0\frac{b_0-a_1b_1}{1-a_1^2}\lambda_w^2 \iff$$

$$\iff (1-a_1c_1)\lambda_e^2 + (1-a_1^2)\lambda_v^2 = (b_0-a_1b_1)\lambda_w^2 b_0$$

$\underline{k\geq 1}$: $$(-a_1)^{k-1}\frac{c_1-a_1}{1-a_1^2}\lambda_e^2 = (-a_1)^k b_0\frac{b_0-a_1b_1}{1-a_1^2}\lambda_w^2 + \lambda_w^2 b_0\sum_{m=1}^{\min\{k,nb\}} b_m(-a_1)^{k-m}$$

$$\Updownarrow$$

$$(a_1-c_1)\lambda_e^2 = b_0a_1(b_0-a_1b_1)\lambda_w^2 + a_1(1-a_1^2)\lambda_w^2 b_0\sum_{m=1}^{\min\{k,nb\}} b_m(-a_1)^{-m}$$

• $\boxed{k\geq nb}$ $\Rightarrow$ ecuații identice:

$$(a_1-c_1)\lambda_e^2 = b_0 a_1(b_0-a_1b_1)\lambda_w^2 + b_0 a_1(1-a_1^2)\lambda_w^2\sum_{m=1}^{nb} b_m(-a_1)^{-m}$$

$$\Downarrow$$

$(nb+1)$ ecuații $(k\in\overline{0,nb})$ și $(nb+2)$ necunoscute: $\{b_0,\dots,b_{nb},\lambda_w^2\}$.
↳ model neunic

$$\sum_{m=1}^{k} b_m(-a_1)^{-m} = \frac{a_1-c_1}{b_0a_1(1-a_1^2)}\cdot\frac{\lambda_e^2}{\lambda_w^2} - \frac{b_0-a_1b_1}{1-a_1^2} = \text{const.}\quad \forall k\in\overline{1,nb}$$

$$\Downarrow$$

$$b_2=b_3=\dots=b_{nb}=0 \Rightarrow \boxed{nb=1}$$

❺ Exerciții rezolvate

Soluție (Exercițiul 1.4)

$k=0$: $(1-a_1c_1)\lambda_e^2 + (1-a_1^2)\lambda_v^2 = (b_0 - a_1 b_1)\lambda_w^2 b_0$

$k=1$: $\dfrac{c_1-a_1}{1-a_1^2}\lambda_e^2 = b_0\dfrac{b_1-a_1b_0}{1-a_1^2}\lambda_w^2 \Leftrightarrow (c_1-a_1)\lambda_e^2 = b_0(b_1-a_1b_0)\lambda_w^2$

$b_0 =$ parametru liber $\neq 0$

$\Downarrow$

$$b_1 = \frac{(c_1-a_1)\lambda_e^2}{b_0\lambda_w^2} + a_1 b_0$$

$$(1-a_1c_1)\lambda_e^2 + (1-a_1^2)\lambda_v^2 = b_0^2\lambda_w^2 - a_1^2 b_0^2\lambda_w^2 - a_1(c_1-a_1)\lambda_e^2$$

$\Downarrow$

$$\left[\begin{array}{l} \lambda_w^2 = \dfrac{1+a_1c_1-a_1^2-a_1c_1}{b_0^2(1-a_1^2)}\lambda_e^2 + \dfrac{1}{b_0^2}\lambda_v^2 = \dfrac{\lambda_e^2+\lambda_v^2}{b_0^2} \\[2ex] b_1 = \dfrac{(c_1-a_1)\lambda_e^2}{b_0\lambda_w^2} + a_1 b_0 \quad ; \quad b_0 \neq 0 \text{ liber.} \end{array}\right.$$

☝ Deoarece $b_0 \neq 0$, are loc transmisia instantanee a zgomotului la ieşire, aşa cum era de aşteptat.

☝ **Examen:** Echivalați un model AR avînd 2 zgomote cu un model ARMA avînd un singur zgomot.